MATH 2D Prep: Integration Techniques

Facts to Know:

1. Substitution Rule:
   - $\int_a^b f(g(x))g'(x)dx = \int_{x=a}^{b} f(u)du = \int_{g(a)}^{g(b)} f(u)du$ $\quad x=a \quad u=g(a)$
   - $u =$ $g(x)$ $\quad du =$ $g'(x)dx$ $\quad \left(\frac{du}{dx} = g'(x)\right)$ $\quad x=b \quad u=g(b)$
2. Integration by part:
   - $\int_a^b f(x)g'(x)dx = \int_{x=a}^{b} u\,dv = uv\Big]_{x=a}^{b} - \int_{x=a}^{b} v\,du = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)dx$
   - $u =$ $f(x)$ $\quad du =$ $f'(x)dx$ $\quad v =$ $g(x)$ $\quad dv =$ $g'(x)dx$

Examples:

1. Evaluate $\int_0^{\pi/4} \cos^2(x)\sin(x)dx$

   $u =$ $\cos(x)$ $\quad du =$ $-\sin(x)dx$ $\quad x=0,\ u=\cos(0)=1$ $\quad x=\pi/4,\ u=\cos(\pi/4)=\frac{\sqrt{2}}{2}$

$$\int_0^{\pi/4} \cos^2(x)\cdot(-du) = \int_{x=0}^{\pi/4} u^2\cdot(-du) = \int_1^{\sqrt{2}/2} -u^2du = -\frac{u^3}{3}\Big]_1^{\sqrt{2}/2}$$
$$= -\frac{1}{3}\left[\frac{1}{2\sqrt{2}} - 1\right] = \frac{1}{3}\left(1 - \frac{1}{2\sqrt{2}}\right)$$

2. Evaluate $\int_0^1 x^2e^x dx$

   $u =$ x^2 $\quad du =$ $2xdx$ $\quad v =$ e^x $\quad dv =$ $e^x dx$,

$$\int_0^1 x^2e^x dx = \int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du$$
$$= x^2e^x\Big]_0^1 - \int_0^1 e^x\cdot 2x\,dx$$
$$= 1^2e^1 - 0 - 2\int_0^1 xe^x dx = e - 2\int_0^1 xe^x dx$$

$$\int_0^1 xe^x dx = \int_0^1 x\,d(e^x) = xe^x\Big]_0^1 - \int_0^1 e^x dx \quad (u=x,\ v=e^x)$$
$$= 1\cdot e^1 - 0 - [e^x]_0^1 = e - 0 - [e-1] = 1$$
$$\Rightarrow \int_0^1 x^2e^x dx = e - 2$$